Solitons and Other Extended Field Configurations

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February 1, 2008

Abstract

Article for the forthcoming Encyclopedia of Mathematical Physics, to be published by Elsevier. Covers kinks & breathers, sigma-models & Skyrmions, abelian-Higgs vortices, monopoles, Yang-Mills instantons, and Q-balls.

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Introduction

A soliton is a localized lump (or string or wall etc) of energy, which can move without distortion, dispersion or dissipation, and which is stable under perturbations (and collisions with other solitons). The word was coined by Zabusky and Kruskal in 1965, to describe a solitary wave with particle-like properties (as in electron, proton etc). Solitons are relevant to numerous areas of physics — condensed-matter, cosmology, fluids/plasmas, biophysics (eg DNA), nuclear physics, high-energy physics etc. Mathematically, they are modelled as solutions of appropriate partial differential equations.

Systems which admit solitons may be classified according to the mechanism by which stability is ensured. Such mechanisms include complete integrability, nontrivial topology plus dynamical balancing, and Q-balls/breathers.

Sometimes the term 'soliton' is used in a restricted sense, to refer to stable localized lumps which have purely elastic interactions: solitons which collide without any radiation being emitted. This is possible only in very special systems, namely those that are completely-integrable. For these systems, soliton stability (and the elasticity of collisions) arise from a number of characteristic properties, including: a precise balance between dispersion and nonlinearity, solvability by the inverse scattering transform from linear data, infinitely many conserved quantities, a Lax formulation (associated linear problem), and Bäcklund transformations. Examples of such integrable soliton systems are the sine-Gordon, Korteweg-deVries and nonlinear Schrödinger equations.

The category of topological solitons is the most varied, and includes such examples as kinks, vortices, monopoles, Skyrmions and instantons. The requirement for these of 'dynamical balancing' can be understood in terms of Derrick's theorem, which provides necessary conditions for a classical field theory to admit static localized solutions. The Derrick argument involves studying what happens to the energy of a field when one changes the scale of space. If one has a scalar field (or multiplet of scalar fields) ϕ, and/or a gauge field $F_{\mu\nu}$, then the static energy E is the sum of terms such as

$$E_0 = \int V(\phi)\, d^n x, \quad E_d = \int T_d(D_j\phi)\, d^n x, \quad E_F = \int F_{jk}F_{jk}\, d^n x,$$

where each integral is over (n-dimensional) space $\mathbf{R}^n$, $D_j\phi$ denotes the covariant spatial derivative of ϕ, and $T_d(\xi_j)$ is a real-valued polynomial of degree d. In particular, for example, we could have $T_2(D_j\phi) = (D_j\phi)(D_j\phi)$: the standard gradient term. Under the dilation $x^j \mapsto \lambda x^j$, these functionals transform as

$$E_0 \mapsto \lambda^{-n}E_0, \quad E_d \mapsto \lambda^{d-n}E_d, \quad E_F \mapsto \lambda^{4-n}E_F.$$

In order to have a static solution (critical point of the static energy functional), one needs to have a zero exponent on λ, and/or a balance between positive and negative exponents. A negative exponent indicates a compressing force (tending to implode a localized lump), whereas a positive exponent indicates an expanding force; so to have a static lump solution, these two forces have to balance each other. For $n = 1$, a system involving only a scalar field, with terms of the form E_0 and E_2, can admit static solitons — for example kinks; the scaling argument implies a Virial Theorem, which in this case says that $E_0 = E_2$. For $n = 2$, one can have a scalar system with only E_2, since in this case the relevant exponent is zero — for example the 2-dimensional sigma-model. Another $n = 2$ example is that of vortices in the Abelian-Higgs model, where the energy contains terms E_0, E_2 and E_F. For $n = 3$, interesting systems have E_2 together with either E_4 (for example Skyrmions) or E_F (for example monopoles). An E_0 term is optional in these cases; its presence affects, in particular, the long-range properties of the solitons. For $N = 4$, one can have instantons in a pure gauge theory (term E_F only).

It should be noted that if there are no restrictions on the fields ϕ and A_j (such as arise, for example, from non-trivial topology), then there is a more obvious mode of instability, which will inevitably be present: $\phi \mapsto \mu\phi$ and/or $A_j \mapsto \mu A_j$, where $0 \leq \mu \leq 1$. In other words, the fields can simply be scaled away altogether, so that the height of the soliton (and its energy) go smoothly to zero. This can be prevented by non-trivial topology.

Another way of preventing solitons from shrinking is to allow the field to have some 'internal' time-dependence, so that it is stationary rather than static. For example, one could allow the complex scalar field ϕ to have the form $\phi = \psi \exp(i\omega t)$, where ψ is independent of time t. This leads to something like a centrifugal force, which can have a stabilizing effect in the absence of Skyrme or magnetic terms. The corresponding solitons are Q-balls.

Kinks and Breathers

The simplest topological solitons are kinks, in systems involving a real-valued scalar field $\phi(x)$ in one spatial dimension. The dynamics is governed by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left[(\phi_t)^2 - (\phi_x)^2 - W(\phi)^2\right],$$

where $W(\phi)$ is some (fixed) smooth function. The system can admit kinks if $W(\phi)$ has at least two zeros, for example $W(A) = W(B) = 0$ with $W(\phi) > 0$ for $A < \phi < B$. Two well-known systems are: sine-Gordon, where $W(\phi) = 2\sin(\phi/2)$, $A = 0$, $B = 2\pi$;

and phi-four, where $W(\phi) = 1 - \phi^2$, $A = -1$, $B = 1$. The corresponding field equations are the Euler-Lagrange equations for $\mathcal{L}$; for example, the sine-Gordon equation is

$$\phi_{tt} - \phi_{xx} + \sin \phi = 0. \tag{1}$$

Configurations satisfying the boundary conditions $\phi \to A$ as $x \to -\infty$ and $\phi \to B$ as $x \to \infty$ are called kinks (and the corresponding ones with $x = \infty$ and $x = -\infty$ interchanged are antikinks). For kink (or antikink) configurations, there is a lower bound, called the Bogomol'nyi bound, on the static energy $E[\phi]$; for kink boundary conditions, we have

$$\begin{aligned}
E[\phi] &= \frac{1}{2} \int_{-\infty}^{\infty} [(\phi_x)^2 + W(\phi)^2] \, dx \\
&= \frac{1}{2} \int_{-\infty}^{\infty} [\phi_x - W(\phi)]^2 \, dx + \int_{-\infty}^{\infty} W(\phi) \phi_x \, dx \\
&\geq \int_A^B W(\phi) \, d\phi,
\end{aligned}$$

with equality if and only if the Bogomol'nyi equation

$$\frac{d\phi}{dx} = W(\phi) \tag{2}$$

is satisfied. A static solution of the Bogomol'nyi equation is a kink solution — it is a static minimum of the energy functional in the kink sector. For example, for the sine-Gordon system we get $E[\phi] \geq 8$, with equality for the sine-Gordon kink

$$\phi(x) = 4 \tan^{-1} \exp(x - x_0);$$

while for the phi-four system we get $E[\phi] \geq 4/3$, with equality for the phi-four kink

$$\phi(x) = \tanh(x - x_0).$$

These kinks are stable topological solitons; the non-trivial topology corresponds to the fact that the boundary value of $\phi(t, x)$ at $x = \infty$ is different from that at $x = -\infty$. With trivial boundary conditions (say $\phi \to A$ as $x \to \pm\infty$) stable static solitons are unlikely to exist, but solitons with periodic time-dependence (which is this context are called breathers) may exist. For example, the sine-Gordon equation and the non-linear Schrödinger equation both admit breathers — but these owe their existence to complete integrability. By contrast, the phi-four system (which is not integrable) does not admit breathers; a collision between a phi-four kink and an antikink (with suitable impact speed) produces a long-lived state which looks like a breather, but eventually decays into radiation.

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In lattice systems, however, breathers are more generic. In a 1-dimensional lattice system, the continuous space $\mathbf{R}$ is replaced by the lattice $\mathbf{Z}$, so $\phi(t, x)$ is replaced by $\phi_n(t)$ where $n \in \mathbf{Z}$. The Lagrangian is

$$L = \frac{1}{2} \sum_n \left[(\dot{\phi}_n)^2 - h^{-2}(\phi_{n+1} - \phi_n)^2 - W(\phi_n) \right];$$

here h is a positive parameter, corresponding to the dimensionless ratio between the lattice spacing and the size of a kink. The continuum limit is $h \to 0$. This system admits kink solutions as in the continuum case; and for h large enough, it admits breathers as well, but these disappear as h becomes small.

Interpreted in three dimensions, the kink becomes a domain wall separating two regions in which the order parameter ϕ takes on distinct values; this has applications in such diverse areas as cosmology and condensed-matter physics.

Sigma-Models and Skyrmions

In a sigma-model or Skyrme system, the field is a map ϕ from space-time to a Riemannian manifold M; usually M is taken to be a Lie group or a symmetric space. The energy density of a static field can be constructed as follows (the Lorentz-invariant extension of this gives a relativistic Lagrangian for fields on space-time). Let ϕ^a be local coordinates on the m-dimensional manifold M, let h_{ab} denote the metric of M, and let x^j denote the spatial coordinates on space $\mathbf{R}^n$. Define an $m \times m$ matrix D by

$$D_a{}^b = (\partial_j \phi^c) h_{ac} (\partial_j \phi^b),$$

where ∂_j denotes derivatives with respect to the x^j. Then the invariants $\mathcal{E}_2 = \mathrm{tr}(D) = |\partial_j \phi^a|^2$ and $\mathcal{E}_4 = \frac{1}{2} \left[(\mathrm{tr}\, D)^2 - \mathrm{tr}(D^2) \right]$ can be terms in the energy density, as well as a zeroth-order term $\mathcal{E}_0 = V(\phi^a)$ not involving derivatives of ϕ. A term of the form $\mathcal{E}_4$ is called a Skyrme term.

The boundary condition on field configurations is that ϕ tends to some constant value $\phi_0 \in M$ as $|x| \to \infty$ in $\mathbf{R}^n$. From the topological point of view, this compactifies $\mathbf{R}^n$ to S^n. In other words, ϕ extends to a map from S^n to M; and such maps are classified topologically by the homotopy group $\pi_n(M)$. For topological solitons to exist, this group has to be non-trivial.

In one spatial dimension ($n = 1$) with $M = S^1$ (say), the expression $\mathcal{E}_4$ is identically zero, and we just have kink-type systems such as sine-Gordon. The simplest two-dimensional example ($n = 2$) is the O(3) sigma model, which has $M = S^2$ with its standard metric. In this system, the field is often expressed as a unit 3-vector field

$\vec{\phi} = (\phi^1, \phi^2, \phi^3)$, with $\mathcal{E}_2 = (\partial_j \vec{\phi}) \cdot (\partial_j \vec{\phi})$. Here the configurations are classified topologically by their degree (or winding number, or topological charge) $N \in \pi_2(S^2) \cong \mathbf{Z}$, which equals

$$N = \frac{1}{4\pi} \int \vec{\phi} \cdot \partial_1 \vec{\phi} \times \partial_2 \vec{\phi} \, dx^1 \, dx^2.$$

It is often convenient, instead of $\vec{\phi}$, to use a single complex-valued function W related to $\vec{\phi}$ by the stereographic projection $W = (\phi^1 + i\phi^2)/(1 - \phi^3)$. In terms of W, the formula for the degree N is

$$N = \frac{i}{2\pi} \int \frac{W_1 \overline{W}_2 - W_2 \overline{W}_1}{(1 + |W|^2)^2} \, dx^1 \, dx^2 \,,$$

and the static energy is (with $z = x^1 + ix^2$)

$$
\begin{aligned}
E &= \int \mathcal{E}_2 \, d^2x \\
&= 8 \int \frac{|W_z|^2 + |W_{\overline{z}}|^2}{(1 + |W|^2)^2} \, d^2x \\
&= 16 \int \frac{|W_{\overline{z}}|^2}{(1 + |W|^2)^2} \, d^2x + 8 \int \frac{|W_z|^2 - |W_{\overline{z}}|^2}{(1 + |W|^2)^2} \, d^2x \\
&= 16 \int \frac{|W_{\overline{z}}|^2}{(1 + |W|^2)^2} \, d^2x + 8\pi N \,.
\end{aligned}
$$

From this, one sees that E satisfies the Bogomol'nyi bound $E \geq 8\pi N$; and that minimal-energy solutions correspond to solutions of the Cauchy-Riemann equations $W_{\overline{z}} = 0$. To have finite energy, $W(z)$ has to be a rational function, and so solutions with with winding number N correspond to rational meromorphic functions $W(z)$, of degree $|N|$. (If $N < 0$, then W is a rational function of $\overline{z}$.) The energy is scale-invariant (conformally-invariant), and consequently these solutions are not solitons — they are not quite stable, since their size is not fixed. Adding terms $\mathcal{E}_4$ and $\mathcal{E}_0$ to the energy density fixes the soliton size, and the resulting two-dimensional Skyrme systems admit true topological solitons.

The three-dimensional case $(n = 3)$, with M being a simple Lie group, is the original Skyrme model of nuclear physics. If $M = SU(2)$, then the integer $N \in \pi_3(SU(2)) \cong \mathbf{Z}$ is interpreted as the baryon number. The (quantum) excitations of the ϕ-field correspond to the pions, whereas the (semi-classical) solitons correspond to the nucleons. This model emerges as an effective theory of QCD, in the limit where the number of colours is large. If we express the field as a function $U(x^j)$ taking values in a Lie group, then $L_j = U^{-1} \partial_j U$ takes values in the corresponding Lie algebra, and $\mathcal{E}_2$ and $\mathcal{E}_4$ take the form

$$
\begin{aligned}
\mathcal{E}_2 &= -\frac{1}{2} \operatorname{tr}(L_j L_j) \\
\mathcal{E}_4 &= -\frac{1}{16} \operatorname{tr} ([L_j, L_k][L_j, L_k]) \,.
\end{aligned}
$$

The static energy density in the basic Skyrme system is the sum of these two terms. The static energy satisfies a Bogomol'nyi bound $E \geq 12\pi^2|N|$, and it is believed that stable solitons (Skyrmions) exist for each value of N. Classical Skyrmions have been investigated numerically; and, for values of N up to about 25, they turn out to resemble polyhedral shells. Comparison with nucleon phenomenology requires semi-classical quantization, and this leads to results which are at least qualitatively correct.

A variant of the Skyrme model is the Skyrme-Faddeev system, which has $n = 3$ and $M = S^2$; the solitons in this case resemble loops which can be linked or knotted, and which are classified by their Hopf number $N \in \pi_3(S^2)$. In this case, the energy satisfies a lower bound of the form $E \geq cN^{3/4}$. Numerical experiments indicate that for each N, there is a minimal-energy solution with Hopf number N, and with energy close to this topological lower bound.

Abelian-Higgs Vortices.

Vortices live in two spatial dimensions, and viewed in three dimensions are string-like; two applications are as cosmic strings and as magnetic flux tubes in superconductors. They occur as static topological solitons in the the abelian Higgs model (or Ginzberg-Landau model), and involve a magnetic field $B = \partial_1 A_2 - \partial_2 A_1$, coupled to a complex scalar field ϕ, on the plane $\mathbf{R}^2$. The energy density is

$$\mathcal{E} = \frac{1}{2}(D_j\phi)(\overline{D_j\phi}) + \frac{1}{2}B^2 + \frac{1}{8}\lambda(1 - |\phi|^2)^2\,, \tag{3}$$

where $D_j\phi := \partial_j\phi - iA_j\phi$, and where λ is a positive constant. The boundary conditions are

$$D_j\phi = 0, \quad B = 0, \quad |\phi| = 1 \tag{4}$$

as $r \to \infty$. If we think of a very large circle C on $\mathbf{R}^2$, so that (4) holds on C, then $\phi|_C$ is a map from the circle C to the circle of unit radius in the complex plane, and therefore it has an integer winding number N. Thus configurations are labelled by this vortex number N.

Note that if $\mathcal{E}$ vanishes, then $B = 0$ and $|\phi| = 1$: the gauge symmetry is spontaneously broken, and the photon 'acquires a mass': this is a standard example of spontaneous symmetry breaking.

The total magnetic flux $\int B\,d^2x$ equals $2\pi N$; a proof of this is as follows. Let θ be the usual polar coordinate around C. Because $|\phi| = 1$ on C, we can write $\phi = \exp[if(\theta)]$ for some function f; this f need not be single-valued, but must satisfy $f(2\pi) - f(0) = 2\pi N$ with N being an integer (in order that ϕ be single-valued). In

fact, this defines the winding number. Now since $D_j\phi = \partial_j\phi - iA_j\phi = 0$ on C, we have

$$A_j = -i\phi^{-1}\partial_j\phi = \partial_j f$$

on C. So, using Stokes' theorem, we get

$$
\begin{aligned}
\int_{\mathbf{R}^2} B\, d^2x &= \int_C A_j\, dx^j \\
&= \int_0^{2\pi} \frac{df}{d\theta}\, d\theta \\
&= 2\pi N\,.
\end{aligned}
$$

If $\lambda = 1$, then the total energy $E = \int \mathcal{E}\, d^2x$ satisfies the Bogomol'nyi bound $E \geq \pi N$; and $E = \pi N$ if and only if a set of partial differential equations (the Bogomol'nyi equations) are satisfied. Since like charges repel, the magnetic force between vortices is repulsive. But there is also a force from the Higgs field, and this is attractive. The balance between the two forces is determined by λ: if $\lambda > 1$, then vortices repel each other; while if $\lambda < 1$, then vortices attract. In the critical case $\lambda = 1$, the force between vortices is exactly balanced, and there exist static multi-vortex solutions. In fact, one has the following: given N points in the plane, there exists an N-vortex solution of the Bogomol'nyi equations (and hence of the full field equations) with ϕ vanishing at the chosen points (and nowhere else). All static solutions are of this form. These solutions cannot, however, be written down explicitly in terms of elementary functions (except of course for $N = 0$).

Monopoles

The abelian Higgs model does not admit 3-dimensional solitons, but a non-abelian generalization does — such non-abelian Higgs solitons are called magnetic monopoles. The field content, in the simplest version, is as follows. First, there is a gauge (Yang-Mills) field $F_{\mu\nu}$, with gauge potential A_μ, and with the gauge group being a simple Lie group G. Secondly, there is a Higgs scalar field ϕ, transforming under the adjoint representation of G (thus ϕ takes values in the Lie algebra of G). For simplicity, take G to be SU(2) in what follows. So we may write $A_\mu = iA_\mu^a\sigma_a$, $F_{\mu\nu} = iF_{\mu\nu}^a\sigma_a$ and $\phi = i\phi^a\sigma_a$, where σ_a are the Pauli matrices. The energy of static ($\partial_0\phi = 0 = \partial_0 A_j$), purely-magnetic ($A_0 = 0$) configurations is

$$E = \int \left[\frac{1}{2}B_j^a B_j^a + \frac{1}{2}(D_j\phi)^a(D_j\phi)^a + \frac{1}{4}\lambda(1 - \phi^a\phi^a)^2 \right] d^3x,$$

where $B_j^a = \frac{1}{2}\epsilon_{jkl}F_{kl}$ is the magnetic field. The boundary conditions are $B_j^a \to 0$ and $\phi^a\phi^a \to 1$ as $r \to \infty$; so ϕ restricted to a large spatial 2-sphere becomes a map from

S^2 to the unit 2-sphere in the Lie algebra $su(2)$, and as such it has a degree $N \in \mathbf{Z}$. An analytic expression for N is

$$\int B_j^a (D_j \phi)^a \, d^3x = 2\pi N. \tag{5}$$

At long range, the field resembles an isolated magnetic pole (a Dirac magnetic monopole), with magnetic charge $2\pi N$. Asymptotically, the SU(2) gauge symmetry is spontaneously broken to U(1), which is interpreted as the electromagnetic gauge group.

In 1974, it was observed pointed out that this system admits a smooth, finite-energy, stable, spherically-symmetric $N = 1$ solution — this is the 't Hooft-Polyakov monopole. There is a Bogomol'nyi lower bound on the energy E: from $0 \leq (B + D\phi)^2 = B^2 + (D\phi)^2 + 2B \cdot D\phi$ we get

$$E \geq 2\pi N + \int \frac{1}{4}\lambda(1 - \phi^a \phi^a)^2 \, d^3x, \tag{6}$$

where use has been made of (5). The inequality (6) is saturated if and only if we go to the Prasad-Sommerfield limit $\lambda = 0$, and the Bogomol'nyi equations

$$(D_j \phi)^a = -B_j^a \tag{7}$$

hold. The corresponding solitons are called BPS monopoles.

The Bogomol'nyi equations (7), together with the boundary conditions described above, form a completely-integrable elliptic system of partial differential equations. For any positive integer N, the space of BPS monopoles of charge N, with gauge freedom factored out, is parametrized by a $(4N - 1)$-dimensional manifold $\mathcal{M}_N$. This is the moduli space of N monopoles. Roughly speaking, each monopole has a position in space (3 parameters) plus a phase (1 parameter), making a total of $4|N|$ parameters; an overall phase can be removed by a gauge transformation, leaving $4|N|-1$ parameters. In fact, it is often useful to retain the overall phase, and to work with the corresponding $4|N|$-dimensional manifold $\widetilde{\mathcal{M}}_N$. This manifold has a natural metric, which corresponds to the expression for the kinetic energy of the system. A point in $\widetilde{\mathcal{M}}_N$ represents an N-monopole configuration, and the slow-motion dynamics of N monopoles corresponds to geodesics on $\widetilde{\mathcal{M}}_N$; this is the geodesic approximation of monopole dynamics.

The $N = 1$ monopole is spherically-symmetric, and the corresponding fields take a simple form; for example, the Higgs field of a 1-monopole located at $r = 0$ is

$$\phi^a = \left[\frac{\coth(2r)}{r} - \frac{1}{2r^2}\right] x^a.$$

For $N > 1$, the expressions tend to be less explicit; but monopole solutions can nevertheless be characterized in a fairly complete way. The Bogomol'nyi equations

(7) are a dimensional reduction of the self-dual Yang-Mills equations in $\mathbf{R}^4$, and BPS monopoles correspond to holomorphic vector bundles over a certain two-dimensional complex manifold ('mini-twistor space'). This leads to various other characterizations of monopole solutions, for example in terms of certain curves ('spectral curves') on mini-twistor space, and in terms of solutions of a set of ordinary differential equations called the Nahm equations. Having all these descriptions enables one to deduce much about the monopole moduli space, and to characterize many monopole solutions. In particular, there are explicit solutions of the Nahm equations involving elliptic functions, which correspond to monopoles with certain discrete symmetries, such as a 3-monopole with tetrahedral symmetry, and a 4-monopole with the appearance and symmetries of a cube.

Yang-Mills Instantons

Consider gauge fields in 4-dimensional Euclidean space $\mathbf{R}^4$, with gauge group G. For simplicity in what follows, take G to be SU(2); one can extend much of the structure to more general groups, for example the simple Lie groups. Let A_μ and $F_{\mu\nu}$ denote the gauge potential and gauge field. The Yang-Mills action is

$$S = -\frac{1}{4} \int \mathrm{tr}\left(F_{\mu\nu}F_{\mu\nu}\right) d^4x, \tag{8}$$

where we assume a boundary condition, at infinity in $\mathbf{R}^4$, such that this integral converges. The Euler-Lagrange equations which describe critical points of the functional S are the Yang-Mills equations

$$D_\mu F_{\mu\nu} = 0. \tag{9}$$

Finite-action Ynag-Mills fields are called instantons. The Euclidean action (8) is used in the path-integral approach to quantum gauge field theory, and therefore instantons are crucial in understanding the path integral.

The dual of the field tensor $F_{\mu\nu}$ is

$$*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}\,;$$

the gauge field is self-dual if $*F_{\mu\nu} = F_{\mu\nu}$, and anti-self-dual if $*F_{\mu\nu} = -F_{\mu\nu}$. In view of the Bianchi identity $D_\mu * F_{\mu\nu} = 0$, any self-dual or anti-self-dual gauge field is automatically a solution of the Yang-Mills equations (9). This fact also follows from the discussion below, where we see that self-dual instantons give local minima of the action.

The Yang-Mills action (and Yang-Mills equations) are conformally invariant; and any finite-action solution of the Yang-Mills equations on R^4 extends smoothly to the

conformal compactification S^4. Gauge fields on S^4, with gauge group SU(2), are classified topologically by an integer N, namely the second Chern number

$$N = c_2 = -\frac{1}{8\pi^2} \int \text{tr}\left(F_{\mu\nu} * F_{\mu\nu}\right) d^4x. \tag{10}$$

From (8) and (10) we get a topological lower bound on the action, as follows:

$$
\begin{aligned}
0 &\leq -\int \text{tr}\left(F_{\mu\nu} - *F_{\mu\nu}\right)\left(F_{\mu\nu} - *F_{\mu\nu}\right) d^4x \\
&= 8S - 16\pi^2 N;
\end{aligned}
$$

and so $S \geq 2\pi^2 N$, with equality if and only if the field is self-dual. If $N < 0$, we get $S \geq 2\pi^2|N|$, with equality if and only if F is anti-self-dual. So the self-dual (or anti-self-dual) fields minimize the action in each topological class.

For the remainder of this section, we restrict to self-dual instantons with instanton number $N > 0$. The space (moduli space) of such instantons, with gauge-equivalence factored out, is a $(8N - 3)$-dimensional real manifold. In principle, all these gauge fields can be constructed using algebraic-geometry (twistor) methods: instantons correspond to holomorphic vector bundles over complex projective 3-space (twistor space). One large class of solutions which can be written out explicitly is as follows; for $N = 1$ and $N = 2$ it gives all instantons, while for $N \geq 3$ it gives a $(5N + 4)$-dimensional subfamily of the full $(8N - 3)$-dimensional solution space. The gauge potentials in this class have the form

$$A_\mu = \text{i}\sigma_{\mu\nu}\partial_\nu \log \phi, \tag{11}$$

where the $\sigma_{\mu\nu}$ are constant matrices (antisymmetric in $\mu\nu$) defined in terms of the Pauli matrices σ_a by

$$
\begin{aligned}
\sigma_{10} &= \sigma_{23} = \frac{1}{2}\sigma_1, \\
\sigma_{20} &= \sigma_{31} = \frac{1}{2}\sigma_2, \\
\sigma_{30} &= \sigma_{12} = \frac{1}{2}\sigma_3.
\end{aligned}
$$

The real-valued function $\phi = \phi(x^\mu)$ is a solution of the 4-dimensional Laplace equation given by

$$\phi(x^\mu) = \sum_{k=0}^{N} \frac{\lambda_k}{(x^\mu - x_k^\mu)(x^\mu - x_k^\mu)},$$

where the x_k^μ are $N+1$ distinct points in $\mathbf{R}^4$, and the λ_k are $N+1$ positive constants: a total of $5N+5$ parameters. It is clear from (11) that the overall scale of ϕ is irrelevant, leaving a $(5N + 4)$-parameter family. For $N = 1$ and $N = 2$, symmetries reduce the

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parameter count further, to 5 and 13 respectively. Although ϕ has poles at the points $x = x_k$, the gauge potentials are smooth (possibly after a gauge transformation).

Finally, it is worth noting that (as one might expect) there is a gravitational analogue of the gauge-theoretic structures described here. In other words, one has self-dual gravitational instantons — these are 4-dimensional Riemannian spaces for which the conformal-curvature tensor (the Weyl tensor) is self-dual, and the Ricci tensor satisfies Einstein's equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$. As before, such spaces can be constructed using a twistor-geometrical correspondence.

Q-Balls

A Q-ball (or nontopological soliton) is a soliton which has a periodic time-dependence in a degree of freedom which corresponds to a global symmetry. The simplest class of Q-ball systems involves a complex scalar field ϕ, with an invariance under the constant phase transformation $\phi \mapsto e^{i\theta}\phi$; the Q-balls are soliton solutions of the form

$$\phi(t, \mathbf{x}) = e^{i\omega t}\psi(\mathbf{x}), \tag{12}$$

where $\psi(\mathbf{x})$ is a complex scalar field depending only on the spatial variables $\mathbf{x}$. The best-known case is the 1-soliton solution

$$\phi(t, x) = a\sqrt{2}\exp(ia^2 t)\operatorname{sech}(ax)$$

of the nonlinear Schrödinger equation $i\phi_t + \phi_{xx} + \phi|\phi|^2 = 0$.

More generally, consider a system (in n spatial dimensions) with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - U(|\phi|),$$

where $\phi(x^\mu)$ is a complex-valued field. Associated with the global phase symmetry is the conserved Noether charge $Q = \int \operatorname{Im}(\bar{\phi}\phi_t)\,d^n x$. Minimizing the energy of a configuration subject to Q being fixed implies that ϕ has the form (12). Without loss of generality, we may take $\omega \geq 0$. Note that $Q = \omega I$, where $I = \int |\psi|^2\,d^n x$. The energy of a configuration of the form (12) is $E = E_q + E_k + E_p$, where

$$
\begin{aligned}
E_q &= \frac{1}{2}\int |\partial_j \psi|^2\,d^n x, \\
E_k &= \frac{1}{2}I\omega^2 = \frac{1}{2}Q^2/I, \\
E_p &= \int U(|\psi|)\,d^n x.
\end{aligned}
$$

Let us take $U(0) = 0 = U'(0)$, with the field satisfying the boundary condition $\psi \to 0$ as $r \to \infty$.

A stationary Q-lump is a critical point of the energy functional $E[\psi]$, subject to Q having some fixed value. The usual (Derrick) scaling argument shows that any stationary Q-lump must satisfy

$$(2 - n)E_q - nE_p + nE_k = 0. \tag{13}$$

For simplicity in what follows, let us take $n \geq 3$. Define $m > 0$ by $U''(0) = m^2$; then, near spatial infinity, the Euler-Lagrange equations give $\nabla^2 \psi - (m^2 - \omega^2)\psi = 0$. So in order to satisfy the boundary condition $\psi \to 0$ as $r \to \infty$, we need $\omega < m$.

It is clear from (13) that if $U \geq \frac{1}{2}m^2|\psi|^2$ everywhere, then there can be no solution. So $K = \min\left[2U(|\psi|)/|\psi|^2\right]$ has to satisfy $K < m^2$. Also, we have

$$E_p = \int U \geq \frac{1}{2}KI = (K/\omega^2)E_k > (K/\omega^2)E_p, \tag{14}$$

where the final inequality comes from (13). As a consequence, we see that ω^2 is restricted to the range

$$K < \omega^2 < m^2. \tag{15}$$

An example which has been studied in some detail is $U(f) = f^2[1 + (1 - f^2)^2]$; here $m^2 = 4$ and $K = 2$, so the range of frequency for Q-balls in this system is $\sqrt{2} < \omega < 2$. The dynamics of Q-balls in systems such as these turns out to be quite complicated.

Further Reading

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See also
Topological defects and their homotopy classification
Instantons in gauge theory
'tHooft-Polyakov monopoles
Vortices